Remark, on Stokes equation

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The Stress of the fluid

Denote u the velocity field et p the pressure field
Then the classical mechanical stress $\boldsymbol{\sigma}^\star$ of the fluid :

$$\boldsymbol{\sigma}^\star(\boldsymbol{u}, p) = 2\mu\mathbb{D}(\boldsymbol{u}) - p\, I_d, \qquad \mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u}) \tag{1}$$

Or in math formulation

$$\boldsymbol{\sigma}^\bullet(\boldsymbol{u}, p) = \mu\nabla\boldsymbol{u} - p\, I_d \tag{2}$$

So $\boldsymbol{\sigma}$ is one of this two stress tensor,

Remark: if $\nabla.\boldsymbol{u} = 0$ then $\nabla.2\mathbb{D}(\boldsymbol{u}) = \nabla.\nabla\boldsymbol{u} + \nabla.{}^t\nabla\boldsymbol{u} = \nabla.\nabla\boldsymbol{u} + \nabla\underbrace{\nabla.\boldsymbol{u}}_{=0} = \nabla.\nabla\boldsymbol{u}$

Stokes equations

In Domain Ω of $\mathbb{R}^d$, find the velocity field $\boldsymbol{u}$ et the pressure field p solution of

$$\begin{aligned} \nabla.\boldsymbol{\sigma}(\boldsymbol{u}, p) &= \boldsymbol{f} \quad (3) \\ -\nabla.\boldsymbol{u} &= 0 \quad (4) \end{aligned}$$

$+$ Boundary condition are defined through the variational form

Where $\boldsymbol{f}$ is the density of force.

Variational form of Stokes equations

In Domain Ω of $\mathbb{R}^d$, find the velocity field $\boldsymbol{u}$ et the pressure field p
Mechanical Variational form of Stokes equation

$$\forall \boldsymbol{v}, q; \quad \int_\Omega 2\mu\mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}) - q\nabla.\boldsymbol{u} - p\nabla.\boldsymbol{v} = \int_\Omega \boldsymbol{f}.\boldsymbol{v} + \int_\Gamma {}^t\boldsymbol{n}\boldsymbol{\sigma}^\star(\boldsymbol{u}, p)\boldsymbol{v}$$

Mathematical Variational form of Stokes equation

$$\forall \boldsymbol{v}, q; \quad \int_\Omega \mu\nabla\boldsymbol{u} : \nabla\boldsymbol{v} - q\nabla.\boldsymbol{u} - p\nabla.\boldsymbol{v} = \int_\Omega \boldsymbol{f}.\boldsymbol{v} + \int_\Gamma {}^t\boldsymbol{n}\boldsymbol{\sigma}^\bullet(\boldsymbol{u}, p)\boldsymbol{v}$$

with Ok, but what is the difference, and remember ${}^t\boldsymbol{n}\boldsymbol{\sigma}^\bullet(\boldsymbol{u}, p)$ are boundary density forces $\boldsymbol{f}_\Gamma$.

Dirichlet Boundary condition

Now $\boldsymbol{u}$ is know on $\Gamma = \partial\Omega$ equal to $\boldsymbol{u}_\Gamma$, so $\boldsymbol{v} = 0$, in this cas the variationnal formulation become

$$\forall \boldsymbol{v} \in (H^1_0)^d, q \in L^2; \quad \int_\Omega 2\mu\mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}) - q\nabla.\boldsymbol{u} - p\nabla.\boldsymbol{v} = \int_\Omega \boldsymbol{f}.\boldsymbol{v}$$

It is easy to see than p will be define though a constant, so problem is well pose in space $((H^1_0)^d, L^2_0)$ (see ...) where $L^2_0 = \{v \in L^2, \int v = 0\}$ this imply at discret level the linear system will be not invertible, but the problem is well pose so we can make a regularization bye adding a small term $-\varepsilon pq$ to the variational form

$$\forall \boldsymbol{v} \in (H^1_0)^d, q \in L^2_0; \quad \int_\Omega 2\mu\mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}) - q\nabla.\boldsymbol{u} - p\nabla.\boldsymbol{v} \boldsymbol{-} \varepsilon pq - = \int_\Omega \boldsymbol{f}.\boldsymbol{v}$$

Warning we are in $(H^1_0)^d \times L^2_0$ not in $(H^1_0)^d \times L^2$. The difference is the following test function $v = 0$, $q = 1$ which imply $\int_\Omega \nabla\boldsymbol{u} + \varepsilon p = 0$, so $0 = 1/\varepsilon \int_\Gamma \boldsymbol{u}.\boldsymbol{n} = -\int_\Omega p$, then $p \in L^2_0$ The regularize problem in $(H^1_0)^d, q \in L^2$ is: Find $\boldsymbol{u}^\varepsilon \in (H^1)^d$, $p^\varepsilon \in L^2$, with $u^\varepsilon_{|\Gamma} = u_\Gamma$ such than

$$\forall \boldsymbol{v} \in (H^1_0)^d, q \in L^2; \quad \int_\Omega 2\mu\mathbb{D}(\boldsymbol{u}^\varepsilon) : \mathbb{D}(\boldsymbol{v}) - q\nabla.\boldsymbol{u}^\varepsilon - p^\varepsilon\nabla.\boldsymbol{v} \boldsymbol{-} \varepsilon p^\varepsilon q - = \int_\Omega \boldsymbol{f}.\boldsymbol{v}$$

and we have $||\boldsymbol{u}^\varepsilon - \boldsymbol{u}||_{H^1} + ||p^\varepsilon - p||_{L^2} \leq C\varepsilon||p||_{L^2}$.

Basic Boundary condition for Stokes equations

Remove or know the boundary term $\int_\Gamma {}^t\boldsymbol{n}\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{v}$

First remark

$$\int_\Gamma {}^t\boldsymbol{n}\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{v} = \int_\Gamma {}^t\boldsymbol{f}_\Gamma \boldsymbol{v}.$$

Where $\boldsymbol{f}_\Gamma$ is the boundary force density (in mechanical formulation) .
On the boundary the trick is to know ${}^t\boldsymbol{f}_\Gamma \boldsymbol{v}$ or to put "$\boldsymbol{v} = 0$" on some component if is $\boldsymbol{u}$ know on this component

So try, with FreeFem++
Execute Stokes-Pipe.edp

Execute Stokes-ext.edp

Navier Boundary condition of Stokes equations

$\boldsymbol{\tau}$ the tangent , $\boldsymbol{n}$ the normal, on Γ, g a given function, remember the boundary force $\boldsymbol{f}_{\Gamma} = {}^t\boldsymbol{n}\boldsymbol{\sigma}(\boldsymbol{u}, p)$.

$$\boldsymbol{u}.\boldsymbol{n} = 0 \quad (5)$$
$$\boldsymbol{f}.\boldsymbol{\tau} = \beta u.\boldsymbol{\tau} + \boldsymbol{g}.\boldsymbol{\tau} \quad (6)$$

This imply add in V.F. in RHS:

$$-\int_{\Gamma} \beta \boldsymbol{u}.\boldsymbol{\tau}\boldsymbol{v}.\boldsymbol{\tau} + \boldsymbol{v}.\boldsymbol{\tau}\boldsymbol{g}.\boldsymbol{\tau} = -\int_{\Gamma} \beta\, {}^t\boldsymbol{u}(\boldsymbol{\tau}\, {}^t\boldsymbol{\tau})\boldsymbol{v} + \boldsymbol{v}.\boldsymbol{\tau}\boldsymbol{g}.\boldsymbol{\tau}$$

Remark, if $\boldsymbol{n} \neq \boldsymbol{e}_i$, change $\boldsymbol{u}.\boldsymbol{n} = 0$ by penalisation we have

$$O = \frac{1}{\epsilon} u.\boldsymbol{n}; \qquad \text{Add to V.F. in RHS} - \int_{\Gamma} \frac{1}{\epsilon}\, {}^t\boldsymbol{u}(\boldsymbol{n}\, {}^t\boldsymbol{n})\boldsymbol{v}$$

Remark, Implementation of Dirichlet Boundary Conditions

Original problem is , Find $\boldsymbol{U} = (\boldsymbol{u_i}) \in \mathbb{R}^n$, such that

$$(AU \quad = B)_i \qquad \text{Dof.} i \notin \Gamma_d \tag{7}$$

$$U_i \quad = G_i = (\Pi_h g)_i \quad \text{Dof.} i \in \Gamma_d \tag{8}$$

where A is the matrices associated to the V.F. , B the RHS of the VF without the Dirichlet Boundary Conditions.
Let us call $\texttt{tgv} = 10^{30}$ a huge value (tres grand valeur), and $I_{\Gamma_d} = ((i \in \Gamma_d)\delta_{ij})$

$$A_{\texttt{tgv}} = A + \texttt{tgv}\ I_{\Gamma_d}, \qquad B_{\texttt{tgv}} = B + \texttt{tgv}\ I_{\Gamma_d}\ G$$

We solve $A_{\texttt{tgv}} U = B_{\texttt{tgv}}$, the approximation is in $O(10^{-30})$, it's better than the number of digits 16, so it's exact not to close to 0.

Execute Stokes-Pipe-Navier.edp Execute Stokes-ext-Navier.edp Execute Stokes-BC.edp

Zero Tangent velocity, and Neumann boundary condition

if $\boldsymbol{u}.\boldsymbol{\tau} = 0$ and at continuous level when $\nabla_{\boldsymbol{\tau}}.u = 0$ and $0 = \nabla.\boldsymbol{u} = \nabla_{\boldsymbol{\tau}}.\boldsymbol{u} + \partial_n \boldsymbol{u}_n$ so $\partial_n u_n = 0$
so in the case

$$\boldsymbol{f}_\Gamma.\boldsymbol{n} = {}^t\boldsymbol{n}\sigma(u,p)\boldsymbol{n} = p$$

and we have the following Boundary condition:

$$p = \boldsymbol{f}_\Gamma.\boldsymbol{n}$$

Curve pipe

Execute Stokes-Pipe-Curve.edp

Incompressible Navier-Stokes with Newton method's

To solve $F(u) = 0$ the Newton's algorithm is

1. u^0 a initial guest
2. do
   - find w^n solution of $DF(u^n)w^n = F(u^n)$
   - $u^{n+1} = u^n - w^n$
   - if($||w^n|| < \varepsilon$) break;

For Navier Stokes problem the algorithm is: $\forall v, q$,

$$F(u,p) = \int_\Omega (u.\nabla)u.v + \nu\nabla u : \nabla v - q\nabla.u - p\nabla.v + BC$$

$$\begin{aligned} DF(u,p)(w,w_p) &= \int_\Omega (w.\nabla)u.v + (u.\nabla)w.v \\ &+ \int_\Omega \nu\nabla w : \nabla v - q\nabla.w - p_w\nabla.v + BC0 \end{aligned}$$

Execute cavityNewton.edp

Execute NSNewtonCyl-100-mpi.edp