To simplify the discussion and presentation, we focus on Poisson Eq. (1.2) associated with zero Dirichlet boundary condition $u|_{\partial\Omega} = u_0 = 0$ for the following theoretical discussions in this article. We can trivially extend the results to linear Eq. (1.1). Now summing up (2.3) over all elements $K \in \mathcal{T}_h$, we have the primal formulation of DDG method and its variations of (1.2) as: find $u \in V_h^k$ such that

$$B_h(u, v) = F(v), \quad \forall v \in V_h^k. \tag{2.10}$$

The bilinear form $B_h(w, v)$ is listed below as,

$$\begin{aligned} B_h(w, v) := &\sum_{K \in \mathcal{T}_h} \int_K \nabla w \cdot \nabla v d\mathbf{x} + \sum_{e \in \mathcal{E}_h^I} \int_e \left(\widehat{w_{\mathbf{n}}} [\![v]\!] ds + \sigma \widetilde{v_{\mathbf{n}}} [\![w]\!] \right) ds \\ &+ \sum_{e \in \mathcal{E}_h^D} \int_e \left(\frac{\beta_0}{h_e} wv - w_{\mathbf{n}} v - \sigma v_{\mathbf{n}} w \right) ds, \end{aligned} \tag{2.11}$$

with the right hand side $F(v)$ given as,

$$F(v) = \int_\Omega f v d\mathbf{x}.$$

with $\widehat{u_{\mathbf{n}}}$ and $\widetilde{v_{\mathbf{n}}}$ defined on the interior element edge $\partial K \in \mathcal{E}_h^I$ as,

$$\begin{cases} \widehat{u_{\mathbf{n}}} = \widehat{\nabla u \cdot \mathbf{n}} = \beta_{0u} \frac{[\![u]\!]}{h_e} + \{\!\{u_{\mathbf{n}}\}\!\} + \beta_1 h_e [\![u_{\mathbf{nn}}]\!], \\ \widetilde{v_{\mathbf{n}}} = \widetilde{\nabla v \cdot \mathbf{n}} = \beta_{0v} \frac{[\![v]\!]}{h_e} + \{\!\{v_{\mathbf{n}}\}\!\} + \beta_1 h_e [\![v_{\mathbf{nn}}]\!]. \end{cases} \tag{2.4}$$

We drop higher order terms and only keep the jump, normal derivative average $\{\!\{u_{\mathbf{n}}\}\!\}$ and second order normal derivative jump $[\![u_{\mathbf{nn}}]\!]$ terms in the numerical flux formula. Notice that

1. *DDG method with interface correction* [18]:

$$\sigma = +1 \text{ in (2.3) with } \widetilde{v_{\mathbf{n}}} = \{\{v_{\mathbf{n}}\}\} \text{ in (2.4)} \tag{2.6}$$

with $\beta_1 = 0$ in the numerical flux $\widehat{u_{\mathbf{n}}}$ of (2.4), the DDG method with interface correction [18] degenerates to the symmetric Interior Penalty method. With $\beta_1 \neq 0$, optimal convergence is observed with a small fixed penalty coefficient applied for all P_k polynomial approximations. For example, we choose fixed $\beta_{0u} = 2$ for all P_k ($k \leq 9$) polynomials in [18]. As is well known, the penalty coefficient (β_{0u} in this case) should be taken large enough, roughly in the scale of k^2 for P_k polynomials to stablize the symmetric Interior Penalty method.